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# The graded identities of upper triangular matrices of size two

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**Abstract**

Let $UT_2$ be the algebra of $2 \times 2$ upper triangular matrices over a field $F$. We first classify all possible gradings on $UT_2$ by a group $G$. It turns out that, up to isomorphism, there is only one non-trivial grading and we study all the graded polynomial identities for such algebra. In case $F$ is of characteristic zero we give a complete description of the space of multilinear graded identities in the language of Young diagrams through the representation theory of the hyperoctahedral group. We finally establish a result concerning the rate of growth of the identities for such algebra by proving that its sequence of graded codimensions has almost polynomial growth. © 2001 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

The algebra $UT_2$ of $2 \times 2$ upper triangular matrices over a field plays an important role in the combinatorial theory of PI-algebras (or algebras with polynomial identity). In general, if $A$ is a PI-algebra, one can attach to $A$ a numerical sequence $c_n(A)$, $n = 1, 2, \ldots$, called the sequence of codimensions of $A$. In [8] it was proved that such sequence is exponentially bounded and its asymptotic behavior has become an interesting invariant of the algebra. Let $Id(A)$ be the $T$-ideal of the free algebra of all polynomial identities of $A$. As a consequence of a theorem of Kemer [5] it follows that if $A$ is an algebra over a field of characteristic zero, then $c_n(A)$ is polynomially bounded if and only if $Id(A) \not\subseteq Id(UT_2)$ and $Id(A) \not\subseteq Id(G)$, where $G$ is

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the infinite dimensional Grassmann algebra. It follows that $UT_2$ has almost polynomial growth of the codimensions, i.e., $c_n(UT_2)$ grows exponentially but for every $T$-ideal $I \underset{\neq}{\supset} Id(UT_2)$, $c_n(I)$ is polynomially bounded.

In this paper, we shall prove that $UT_2$ has a similar property regarded as a graded algebra. From the theory of Kemer concerning the structure of the varieties of associative algebras it turns out that a prominent role is played by the superalgebras (or $\mathbf{Z}_2$-graded algebras) and their identities. Recall that an algebra $A$ is graded by a group $G$ if $A = \bigoplus_{g \in G} A_g$ where for $g \in G$, $A_g$ is a subspace of $A$ and $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. In this paper, we shall first classify all possible gradings on the algebra $UT_2$ by a group $G$. We shall prove that, up to isomorphism, only two gradings are allowed on $UT_2$: the trivial grading $UT_2 = (UT_2)_1$, and the canonical $\mathbf{Z}_2$-grading given by $UT_2 = (UT_2)_0 \oplus (UT_2)_1$, where $(UT_2)_0 = Fe_{11} + Fe_{22}$ and $(UT_2)_1 = Fe_{12}$ with the $e_{ij}$'s the usual matrix units.

We shall then extensively study the ideal of graded identities of $UT_2$ in characteristic zero. Recall that a polynomial in two distinct sets of non-commutative variables $f(y_1, \dots, y_m, z_1, \dots, z_m)$ is a graded identity for the superalgebra $A = A_0 \oplus A_1$ if $f(a_1, \dots, a_m, b_1, \dots, b_m) = 0$ for all $a_1, \dots, a_m \in A_0$, $b_1, \dots, b_m \in A_1$. By exploiting a well known duality between $\mathbf{Z}_2$-gradings and automorphisms of order two of an algebra $A$, one has naturally the notion of multilinear graded polynomial of degree $n$ in $y_1, z_1, \dots, y_n, z_n$ (in each monomial either $y_i$ or $z_i$ appears, for all $i = 1, \dots, n$); we write $P_n^{\mathrm{gr}}$ for the space of such polynomials.

In this paper we shall first compute a set of generators for the ideal $Id^{\mathrm{gr}}(UT_2)$ of graded identities of $UT_2$, then we shall describe the space of multilinear graded identities of $UT_2$ of any degree through the representation theory of the hyperoctahedral group $H_n$ as follows: the group $H_n$ acts naturally on the space $P_n^{\mathrm{gr}} \cap Id^{\mathrm{gr}}(UT_2)$ of multilinear graded identities of degree $n$; by complete reducibility, the space $P_n^{\mathrm{gr}} \cap Id^{\mathrm{gr}}(UT_2)$ splits into the direct sum of irreducibles and we shall compute for each irreducible the corresponding multiplicity.

For a superalgebra $A$, the sequence

$$c_n^{\mathrm{gr}}(A) = \dim \frac{P_n^{\mathrm{gr}}}{P_n^{\mathrm{gr}} \cap Id^{\mathrm{gr}}(UT_2)}, \quad n = 1, 2, \dots$$

is called the sequence of graded codimensions of $A$. As a consequence of our description we shall compute the asymptotic behavior of the sequence $c_n^{\mathrm{gr}}(UT_2)$. It turns out that the algebra $UT_2$ with the canonical grading has almost polynomial growth of the graded codimensions; in fact we shall prove that $c_n^{\mathrm{gr}}(UT_2)$ has exponential growth equal to two and any ideal of graded identity properly containing $Id^{\mathrm{gr}}(UT_2)$ has polynomially bounded growth of the graded codimensions.

## 2. Gradings on $UT_2$

Let $A$ be an associative algebra over the field $F$ and let $G$ be a group. Recall that a $G$-grading on $A$ is a decomposition of $A$ into the direct sum of subspaces $A = \bigoplus_{g \in G} A_g$

such that $A_gA_h \subseteq A_{gh}$ for all $g, h \in G$. We say that an element $a \in A$ is homogeneous of degree $g$ if $a \in A_g$. In case $A_g = 0$ for all $g \neq 1$ then we say that $A$ has trivial grading.

We shall next give a complete description of all gradings on $UT_2$, the algebra of $2 \times 2$ upper triangular matrices over $F$. Let $G$ be an arbitrary group, we say that $UT_2$ has the canonical $G$-grading if there exists $g \in G, g \neq 1$ such that $UT_2 = (UT_2)_1 \oplus (UT_2)_g$, where $(UT_2)_1 = Fe_{11} + Fe_{22}$ and $(UT_2)_g = Fe_{12}$. We have the following:

**Theorem 1.** *Any $G$-grading on $UT_2$ is, up to isomorphism, either trivial or canonical.*

**Proof.** Write $A = UT_2$ and let $e \in G$ be the unit element of G. If $\dim A_e = 3$ then $A$ has trivial grading. Hence we may assume that $\dim A_e \leqslant 2$.

Suppose first that $\dim A_e = 2$. We may clearly assume that $e_{11} + e_{22}$ and $ae_{11} + be_{12}$ form a basis of $A_e$ over $F$, for suitable $a, b \in F$. Since $\dim_F UT_2 = 3$, there exists $g \in G$ such that $\dim A_g = 1$ and let $A_g = F(a'e_{11} + b'e_{12} + c'e_{22})$. In case $a = 0$, then the inclusions $A_gA_e \subseteq A_g$ and $A_eA_g \subseteq A_g$ lead to $a' = c' = 0$. Hence $A_g = Fe_{12} \subseteq A_e$, a contradiction. Thus $a \neq 0$. It follows that the elements $e_{11} + be_{12}$ and $e_{22} - be_{12}$ span $A_e$ over $F$. Suppose first that $b \neq 0$. Since $(a'e_{11} + b'e_{12} + c'e_{22})(e_{11} + be_{12}) = a'(e_{11} + be_{12}) \in A_g \cap A_e = 0$ we obtain that $a' = 0$. Similarly, by multiplying $b'e_{12} + c'e_{22}$ on the left by $e_{22} - be_{12}$, we obtain $c' = 0$. Hence $A_g = Fe_{12}, A_e = F(e_{11} + e_{22}) + F(e_{11} + be_{12})$ and $A_e \oplus A_g$ is isomorphic to $UT_2$ with the canonical $G$-grading. In case $b = 0, A_e = Fe_{11} + Fe_{22}$ and it easily follows that $A_g = Fe_{12}$. Thus we are done in this case too.

Suppose now that $\dim A_e = 1$ that is $A_e = F(e_{11} + be_{12})$. So either $UT_2 = A_e \oplus A_g \oplus A_h$ where $\dim A_g = \dim A_h = 1$ or $UT_2 = A_e \oplus A_g$ with $\dim A_g = 2$. Let $UT_2 = A_e \oplus A_g \oplus A_h$ and suppose first that $gh \neq e$. Then $A_gA_h = 0$ and, in case $g^2 \neq e$ and $h^2 \neq e$ we get that $A_g \oplus A_h$ is a two-dimensional nilpotent ideal of $UT_2$, contradicting the fact that $\dim J = 1$,where $J$ is the Jacobson radical of $UT_2$. Hence either $g^2 \neq e, h^2 = e$ or $g^2 = h^2 = e$. In the first case one easily gets that $A_g = J$ and let $A_h = F(ae_{11} + be_{12} + ce_{22})$. From $A_gA_h = A_hA_g = 0$ we easily obtain $a = c = 0$. Hence $A_h = A_g$, a contradiction. In case $g^2 = h^2 = e$, since $A_g = Fu, A_h = Fv$, where $u^2 = v^2 = 1$, we get that $0 \neq uv \in A_gA_h$, a contradiction.

Suppose now that $gh = e$. If $g^3 \neq e$ then $g^2 \neq g^{-1}, (g^{-1}) \neq g$ hence $(A_g)^2 = (A_h)^2 = 0$, a contradiction. In case $g^3 = e$ we obtain that $A_g = Fa$ where $a = \alpha e_{11} + e_{22}$ with $\alpha$ a third root of the unity. Then we would get that $\dim(Fa + Fa^2 + Fe) = 2$, a contradiction. We are left with $UT_2 = A_e \oplus A_g$ and $\dim A_g = 2$. If $g^2 \neq e$ it follows that $A_g$ is a nilpotent ideal, hence $A_g \subseteq J$, and this is a contradiction. Thus $g^2 = e$. Since $A_gA_g \subseteq A_e$ we easily obtain a contradiction also in this case. □

In case of a finite abelian group $G$ all possible $G$-gradings of $UT_n(F)$ the algebra of $n \times n$ upper triangular matrices are described in [9] provided that $F$ is an algebraically closed field of characteristic zero.

## 3. Graded cocharacters and codimensions

Throughout $F$ is a field of characteristic zero and $A = A_0 \oplus A_1$ a $\mathbf{Z}_2$-graded algebra. Let $F\langle X \rangle$ be the free associative algebra over the countable set $X = \{x_1, x_2, \ldots\}$. If we write $X=Y\cup Z$ where $Y=\{y_1,y_2,\ldots\}, Z=\{z_1,z_2,\ldots\}$ and $Y\cap Z=\emptyset$, then $F\langle X\rangle$ has a natural structure of free superalgebra on $X$ by assuming that the variables $y_i\in Y$ and $z_i\in Z$ are homogeneous of degree zero and one, respectively. More precisely, if $\mathscr{F}_0$ is the subspace of $F\langle X\rangle$ generated by all monomials in the variables of $X$ having even degree in the variables of $Z$ and $\mathscr{F}_1$ the subspace of $F\langle X\rangle$ generated by all monomials of odd degree in $Z$, then $F\langle X\rangle=\mathscr{F}_0\oplus\mathscr{F}_1$ is the induced grading; moreover for any superalgebra $A=A_0\oplus A_1$, any map $Y\cup Z\to A$ preserving the grading can be extended in a unique way to a homomorphism of superalgebras $F\langle X\rangle\to A$.

It is well known, that, for any $F$-algebra $A$, there is a duality between $\mathbf{Z}_2$-gradings of $A$ and $\phi$-actions where $\phi\in\mathrm{Aut}(A)$ is an automorphism of order two: in fact if $A=A_0\oplus A_1$ is a $\mathbf{Z}_2$-grading on $A$, then $\phi:A\to A$ such that $\phi(a_0+a_1)=a_0-a_1$ is the defined automorphism. Viceversa, if $\phi$ is an automorphism of order two then $A=A_0\oplus A_1$ where $A_0=\{a\in A|\phi(a)=a\}$ and $A_1=\{a\in A|\phi(a)=-a\}$. Set $\bar{x}_i=y_i+z_i$ and $\bar{x}_i^{\phi}=y_i-z_i,\quad i=1,2,\ldots$ and require that $\phi$ acts as an automorphism of order two on $F\langle X\rangle$. Then $F\langle X\rangle$ becomes the free algebra on $X$ with $\phi$-action. This means that if $A$ is any algebra with an automorphism $\phi,\phi^2=1$, then any map $f:\{\bar{x}_1,\bar{x}_2,\ldots\}\to A$ extends uniquely to a homomorphism $\bar{f}:F\langle X\rangle\to A$ such that $\bar{f}(\bar{x}_i^{\phi})=f(\bar{x}_i)^{\phi}$.

The notion of $\phi$-identity or graded identity for a superalgebra $A=A_0\oplus A_1$, are the obvious ones: $f(y_1,\ldots,y_n,z_1,\ldots,z_n)=f(\bar{x}_1,\bar{x}_1^{\phi}\ldots,\bar{x}_n,\bar{x}_n^{\phi})$ is a graded identity (or a $\phi$ -identity) for $A$ if $f(a_1,\ldots,a_n,b_1,\ldots,b_n)=0$ for all $a_1,\ldots,a_n\in A_0,b_1,\ldots,b_n\in A_1$ ($f(c_1,c_1^{\phi}\ldots,c_n,c_n^{\phi})=0$ for all $c_1,\ldots,c_n\in A$, respectively). Let $Id^{\mathrm{gr}}(A)=Id^{\phi}(A)$ be the ideal of $F\langle X\rangle$ of graded identities (or $\phi$ -identities) of $A$. It is clear that $Id^{\mathrm{gr}}(A)$ is a $T_2$-ideal of $F\langle X\rangle$ i.e., an ideal invariant under all endomorphisms of $F\langle X\rangle$ preserving the grading. Also, since char $F=0$, by standard arguments it is well known that $Id^{\mathrm{gr}}(A)$ is completely determined by its multilinear polynomials.

Let $P_n^{\phi}$ be the vector space of multilinear $\phi$-polynomials of degree $n$ in $\bar{x}_1,\bar{x}_1^{\phi},\ldots,\bar{x}_n,\bar{x}_n^{\phi}$. It follows that

$$P_n^{\phi}=P_n^{\mathrm{gr}}=\mathrm{Span}_F\,\{w_{\sigma(1)}\cdots w_{\sigma(n)}\,|\,\sigma\in S_n,w_i=y_i\text{ or }w_i=z_i,i=1,\ldots,n\}.$$

Let $H_n$ be the hyperoctahedral group of degree $n$. Recall that $H_n=\mathbf{Z}_2\sim S_n$ is the wreath product of $\mathbf{Z}_2=\{1,\phi\}$ and $S_n$, the symmetric group of degree $n$. The space $P_n^{\mathrm{gr}}$ has a natural structure of left $H_n$-module induced by defining for $h=(a_1,\ldots,a_n;\sigma)\in H_n$, $hy_i=y_{\sigma(i)},\ hz_i=z_{\sigma(i)}^{\phi}=\pm z_{\sigma(i)}$ (see [2]).

Let $P_n^{\mathrm{gr}}(A)=(P_n^{\mathrm{gr}}/P_n^{\mathrm{gr}}\cap Id^{\mathrm{gr}}(A))$ be the space of multilinear elements of degree $n$ in the relatively free algebra $F\langle X\rangle/Id^{\mathrm{gr}}(A)$. Since $P_n^{\mathrm{gr}}\cap Id^{\mathrm{gr}}(A)$ is a subspace invariant under the above action we can view $P_n^{\mathrm{gr}}(A)$ as an $H_n$-module; let $\chi_n^{\mathrm{gr}}(A)$ be its character. The sequence

$$c_n^{\mathrm{gr}}(A)=\chi_n^{\mathrm{gr}}(A)(1)=\dim_F P_n^{\mathrm{gr}}(A),\quad n=1,2,\ldots$$

is called the sequence of graded-codimensions of $A$.

Recall that there is a one-to-one correspondence between irreducible $H_n$-characters and pairs of partitions $(\lambda,\mu)$, where $\lambda\vdash r,\ \mu\vdash n-r$, for all $r=0,1,\ldots,n$. If $\chi_{\lambda,\mu}$

denotes the irreducible $H_n$-character corresponding to $(\lambda,\mu)$ then we can write

$$\chi_n^{\mathrm{gr}}(A)=\sum_{r=0}^{n}\sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}\chi_{\lambda,\mu},$$

where $m_{\lambda,\mu}\geqslant 0$ are the corresponding multiplicities.

For fixed $r\in\{0,\dots,n\}$, let

$$P_{r,n-r}=\mathrm{Span}_F\{w_{\sigma(1)}\cdots w_{\sigma(n)}\mid \sigma\in S_n, w_i=y_i \text{ for } i=1,\dots,r, \text{ and } w_i=z_i \text{ for } i=r+1,\dots,n\}$$

be the space of multilinear polynomials in the variables $y_1,\dots,y_r$, $z_{r+1},\dots,z_n$. By the multihomogeneity of $T_2$-ideals, in order to study $P_n^{\mathrm{gr}}(A)$ it is enough to study

$$P_{r,n-r}^{\mathrm{gr}}(A)=\frac{P_{r,n-r}}{P_{r,n-r}\cap Id^{\mathrm{gr}}(A)}$$

for all $r=0,\dots,n$.

If we let $S_r$ act on the variables $y_1,\dots,y_r$ and $S_{n-r}$ act on the variables $z_{r+1},\dots,z_n$, we obtain an action of $S_r\times S_{n-r}$ on $P_{r,n-r}$ and $P_{r,n-r}^{\mathrm{gr}}(A)$ becomes a left $S_r\times S_{n-r}$-module.

Let $\chi_{r,n-r}(A)$ be its character. It is well known that the irreducible $S_r\times S_{n-r}$ characters are obtained by taking the outer tensor product of $S_r$ and $S_{n-r}$ irreducible characters, respectively. Then, we can write

$$\chi_{r,n-r}(A)=\sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}(\chi_\lambda\otimes\chi_\mu),$$

where $\chi_\lambda$ (respectively, $\chi_\mu$) denotes the irreducible $S_r$-character (respectively $S_{n-r}$-character) and $m_{\lambda,\mu}\geqslant 0$ are the corresponding multiplicities.

The relation between $H_n$-characters and the $S_r\times S_{n-r}$-characters is given for instance in [1, Theorem 1.3] as follows: for any superalgebra $A$ we have

$$\chi_n^{\mathrm{gr}}(A)=\sum_{r=0}^{n}\sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}\chi_{\lambda,\mu} \quad\text{and}\quad \chi_{r,n-r}(A)=\sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}(\chi_\lambda\otimes\chi_\mu)$$

for all $r\leqslant n$. Moreover

$$c_n^{\mathrm{gr}}(A)=\sum_{r=0}^{n}\binom{n}{r}\dim_F P_{r,n-r}^{\mathrm{gr}}(A).$$

## 4. Graded identities of $UT_2$

Throughout this section $A=UT_2$ is the algebra of $2\times 2$ upper triangular matrices with canonical $\mathbf{Z}_2$-grading. It is easy to verify that $z_1z_2\equiv 0$ and $[y_1,y_2]\equiv 0$ are graded identities of $A$. We wish to show that these two identities generate $Id^{\mathrm{gr}}(A)$ as a $T_2$-ideal.

Now, if $S \subseteq F\langle X\rangle$, is any set of polynomials, we denote by $\langle S\rangle_{T_2}$ the $T_2$-ideal of $F\langle X\rangle$ generated by $S$. We start with the following:

**Remark 1.** For any variable $x = y + z$, $z_1xz_2 \in \langle z_1z_2, [y_1, y_2]\rangle_{T_2}$.

**Proof.** Write $F\langle X\rangle = \mathscr{F}_0 \oplus \mathscr{F}_1$ as in the previous section. Since $z_1y \in \mathscr{F}_1$ it follows that $z_1yz_2 \in \langle z_1z_2\rangle_{T_2}$. Hence $z_1(y + z)z_2, \in \langle z_1z_2\rangle_{T_2}$ and, so, $z_1xz_2 \in \langle z_1z_2\rangle_{T_2}$. $\square$

In Theorem 2 we exhibit a set of generators for $Id^{\mathrm{gr}}(A)$.

**Theorem 2.** *The identities* $z_1z_2 \equiv 0$ and $[y_1, y_2] \equiv 0$ *generate* $Id^{\mathrm{gr}}(UT_2)$ *as a* $T_2$*-ideal.*

**Proof.** Write $I = \langle z_1z_2, [y_1, y_2]\rangle_{T_2}$ and let $f(y_1,\dots,y_t,z_1,\dots,z_t)$ be a multilinear polynomial in $Id^{\mathrm{gr}}(A)$. We wish to show that, modulo $I$, $f$ is the zero polynomial. From the previous remark it is clear that we can write $f(y_1,\dots,y_t,z_1,\dots,z_t) = f_1(y_1,\dots,y_s) + f_2(z, y_1,\dots,y_s)$ (mod $I$) and, by the multihomogeneity of $T_2$-ideals, it follows that $f_1$ and $f_2$ are both identities of $A$. Since $[y_1, y_2] \in I$, we obtain that $f_1 = \alpha y_1 \cdots y_s$. But then, by substituting $y_1 = \cdots = y_s = e_{11}$ we obtain $\alpha = 0$ and, so, $f_1 = 0$ (mod $I$). Write $f_2 = \sum \alpha y_{i_1} \cdots y_{i_t} z y_{j_1} \cdots y_{j_{n-t}}$ where $i_1 < \cdots < i_t$ and $j_1 < \cdots < j_{n-t}$. Fix one non-zero monomial of $f_2$, let it be $\alpha y_1 \cdots y_s z y_{s+1} \cdots y_n$. By substituting $y_1 = \cdots = y_s = e_{11}, y_{s+1} = \cdots = y_n = e_{22}$, $z = e_{12}$ we get $f = \alpha e_{12}$. Hence $\alpha = 0$, a contradiction. It follows that $f_2 = 0$ (mod $I$) and we are done. $\square$

For any partition $\lambda \vdash n$ let $T_\lambda$ be a Young tableau of shape $\lambda$ and $e_{T_\lambda}$ the corresponding minimal essential idempotent of the group algebra $FS_n$. Recall that $e_{T_\lambda} = \sum_{\substack{\sigma\in R_{T_\lambda}\\ \tau\in C_{T_\lambda}}} (\mathrm{sgn}\,\tau)\sigma\tau$ where $R_{T_\lambda}$ and $C_{T_\lambda}$ are the subgroups of row and column permutations of $T_\lambda$, respectively.

Let $\lambda \vdash r$, $\mu \vdash n - r$ and let $W_{\lambda,\mu}$ be a left irreducible $S_r \times S_{n-r}$-module. It is well known that if $T_\lambda$ is a tableau of shape $\lambda$ and $T_\mu$ a tableau of shape $\mu$, then $W_{\lambda,\mu} \cong F(S_r \times S_{n-r})e_{T_\lambda}e_{T_\mu}$ where $S_r$ and $S_{n-r}$ act on disjoint sets of integers.

We can now write the explicit decomposition of the $n$th graded cocharacter of $A$ into irreducibles. For a partition $\lambda \vdash n$ we denote by $h(\lambda)$ the height of the diagram associated to $\lambda$.

**Theorem 3.** *Let* $\chi_n^{\mathrm{gr}}(UT_2) = \sum_{r=0}^{n} \sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}\chi_{\lambda,\mu}$ *be the nth graded cocharacter of* $UT_2$ *with canonical* $\mathbf{Z}_2$*-grading. Then*
(1) $m_{\lambda,\mu} = q + 1$ *if* $\lambda = (p + q, p), \mu = (1)$;
(2) $m_{(n),\emptyset} = 1$;
(3) $m_{\lambda,\mu} = 0$ *in all other cases.*

**Proof.** The proof here is similar to that of Lemma 1 in [7]. We reproduce it for the convenience of the reader. Since $\dim A_0 = 2$ and $\dim A_1 = 1$, any polynomial alternating on three even variables or in two odd variables vanishes in $A$. From the general form of the elements $e_{T_\lambda}e_{T_\mu}$ it follows that $m_{\lambda,\mu} = 0$ if either $h(\lambda) > 2$ or $h(\mu) > 1$. Moreover by

Remark 1, for every variable $x$, $z_1xz_2 \in Id^{\mathrm{gr}}(A)$ and this implies that $m_{\lambda,\mu}=0$ whenever $|\mu| \geqslant 2$. So let us assume that $|\mu| \leqslant 1$ and suppose first that $\mu=\emptyset$. Then, $[y_1,y_2]\equiv 0$ on $A$ implies that $y_1y_2\cdots y_n$ is a basis of $P_{n,0}$ (mod $Id^{\mathrm{gr}}(A)$). Hence $m_{(n),\emptyset}=1$ and $m_{\lambda,\emptyset}=0$ if $\lambda \neq (n)$.

Suppose now that $h(\lambda)\leqslant 2$ and $\mu=(1)$. Let $\lambda=(p+q,p)$, $p\geqslant 0$, $q\geqslant 0$ and $\mu=(1)$. We want to prove that $m_{\lambda,\mu}=q+1$.

For every $i=0,\dots,q$ define the following two tableaux:

$T_\lambda^{(i)}$ is the tableau

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $i+1$ | $i+2$ | $\cdots$ | $i+p$ | $1$ | $2$ | $\cdots$ | $i$ | $i+2p+2$ | $\cdots$ | $n$ |
| $i+p+2$ | $i+p+3$ | $\cdots$ | $i+2p+1$ | | | | | | | |

and

$$T_\mu^{(i)} = \boxed{i+p+1}.$$

We associate to $T_\lambda^{(i)}$ and $T_\mu^{(i)}$ the polynomial

$$a_{p,q}^{(i)}(y_1,y_2,z) = y_1^i\underbrace{\bar{y}_1\cdots\tilde{y}_1}_{p}z\underbrace{\bar{y}_2\cdots\tilde{y}_2}_{p}y_1^{q-i},$$

where $-,\tilde{}$ mean alternation on the corresponding elements.

Notice that the polynomial $a_{p,q}^{(i)}$ is obtained from the essential idempotent corresponding to the pair of tableaux $(T_\lambda^{(i)},T_\mu^{(i)})$ by identifying all the elements in each row of $\lambda$.

We shall prove that (mod $Id^{\mathrm{gr}}(A)$) the $q+1$ polynomials $a_{p,q}^{(i)}(y_1,y_2,z)$, $i=0,\dots,q$, are linearly independent over $F$.

Suppose not. Let $\sum_{i=0}^q \alpha_i a_{p,q}^{(i)}(y_1,y_2,z)=0$ (mod$^{\mathrm{gr}}(A)$) and let $t=\max\{i:\alpha_i\neq 0\}$. Then $\alpha_t a_{p,q}^{(t)}+\sum_{i<t}\alpha_i a_{p,q}^{(i)}=0$ (mod $Id^{\mathrm{gr}}(A)$).

If we substitute $y_1$ with $y_1+y_3$, we obtain

$$\begin{aligned}&\alpha_t(y_1+y_3)^t\overline{(y_1+y_3)}\cdots\widetilde{(y_1+y_3)}z\bar{y}_2\cdots\tilde{y}_2(y_1+y_3)^{q-t}\\ &\quad+\sum_{i<t}\alpha_i(y_1+y_3)^i\overline{(y_1+y_3)}\cdots\widetilde{(y_1+y_3)}z\bar{y}_2\cdots\tilde{y}_2(y_1+y_3)^{q-i}\\ &\quad=0 \ (\mathrm{mod}\ Id^{\mathrm{gr}}(A)).\end{aligned}$$

Let us consider the homogeneous component of degree $t+p$ in $y_1$ and of degree $q-t$ in $y_3$. If we make the substitution $y_1=e_{11}$, $y_2=y_3=e_{22}$ and $z=e_{12}$, we obtain $\alpha_te_{12}=\alpha_t=0$, a contradiction. Hence the polynomials $a_{p,q}^{(i)}$, $i=0,\dots,q$ are linearly independent (mod $Id^{\mathrm{gr}}(A)$).

Notice that, for all $i$, $e_{T_\lambda^{(i)}}e_{T_\mu^{(i)}}(y_1,\dots,y_{n-1},z)$ is the complete linearization of $a_{p,q}^{(i)}(y_1,y_2,z)$. It follows that the polynomials $e_{T_\lambda^{(i)}}e_{T_\mu^{(i)}}$, $i=0,\dots,q$, are linearly independent (mod $Id^{\mathrm{gr}}(A)$) and this implies that $m_{\lambda,\mu}\geqslant q+1$.

Let now, $T_\lambda$ and $T_\mu$ be any two tableaux and $f = e_{T_\lambda} e_{T_\mu}(y_1,\ldots,y_{n-1},z)$ the corresponding polynomial.

If $f \notin \langle z_1z_2,[y_1,y_2]\rangle_{T_2}$, then any two alternating variables in $f$ must lie on different sides of $z$. Since $f$ is a linear combination (mod $Id^{\mathrm{gr}}(A)$) of polynomials each alternating on $p$ pairs of $y_i$'s, we obtain that $f$ is a linear combination of the polynomials $e_{T_\lambda^{(i)}} e_{T_\mu^{(i)}}$, $i = 0,\ldots,q$. Hence $m_{\lambda,\mu} = q+1$. $\square$

## 5. Some numerical invariants

For an algebra $A$ we shall denote by $Id(A)$ the $T$-ideal of the free algebra $F\langle X\rangle$ of (ordinary) polynomial identities of $A$. If we denote with $P_n$ the space of multilinear polynomials in $x_1,\ldots,x_n$, then $c_n(A) = \dim_F P_n/P_n \cap Id(A)$, $n = 1,2,\ldots$, is the sequence of codimensions of $A$.

Suppose now that $A$ is a superalgebra satisfying a non-trivial polynomial identity. As we mentioned before, the graded identities coincide with the $\phi$-identities where $\phi$ is an automorphism of order two. But then by [3] it follows that

$$c_n(A) \leqslant c_n^{\mathrm{gr}}(A) \leqslant 2^n c_n(A).$$

In order to capture the exponential behavior of the sequence of graded codimensions we then define $\exp^{\mathrm{gr}}(A) = \lim_{n\to\infty} \sup \sqrt{c_n^{\mathrm{gr}}(A)}$.

Another numerical sequence that can be attached to a superalgebra is given by the sequence of colengths: if $\chi_n^{\mathrm{gr}}(A) = \sum_{r=0}^{n} \sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}\chi_{\lambda,\mu}$ is the decomposition of the $n$th graded cocharacter of $A$, then one defines the $n$th graded-colength of $A$ as

$$\mathrm{l}_n^{\mathrm{gr}}(A) = \sum_{r=0}^{n} \sum_{\substack{\lambda\vdash r\\ \mu\vdash n-r}} m_{\lambda,\mu}.$$

In the next corollary we shall compute $\mathrm{Exp}^{\mathrm{gr}}(UT_2)$ and $\mathrm{l}_n^{\mathrm{gr}}(UT_2)$ for all $n \geqslant 1$, in case $UT_2$ is endowed with the canonical $\mathbf{Z}_2$-grading. We need to introduce some notation. For a partition $\lambda \vdash n$, let $d_\lambda = \chi_\lambda(1)$ be the degree of the irreducible $S_n$-character $\chi_\lambda$. If $d_{\lambda,\mu} = \chi_{\lambda,\mu}(1)$ is the degree of the irreducible $H_n$-character corresponding to the pair $(\lambda,\mu)$, where $\lambda \vdash r$, $\mu \vdash n-r$, then it is well known that $d_{\lambda,\mu} = \binom{n}{r} d_\lambda d_\mu$.

**Corollary 1.**
(1) $\mathrm{Exp}^{\mathrm{gr}}(UT_2) = 2$.
(2) *For all* $n \geqslant 1$, $\mathrm{l}_n(UT_2) = \begin{cases} \dfrac{n^2-2n+9}{4} & \textit{if } n \textit{ is odd},\\ \dfrac{n^2-2n+8}{4} & \textit{if } n \textit{ is even}.\end{cases}$

**Proof.** The codimensions of $UT_2$ are known and can be deduced for instance from [6]. It follows that $c_n(UT_2) = \alpha n^t 2^n$, for some constants $\alpha, t$. On the other hand, by recalling that the multiplicities $m_{\lambda,\mu}$ are polynomially bounded, we get

$$c_n(UT_2) \leqslant c_n^{\mathrm{gr}}(UT_2) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu} d_{\lambda,\mu} \leqslant \alpha n^v \sum_{\substack{|\lambda|+|\mu|=n \\ h(\lambda)\leqslant 2,\, \mu_1 \leqslant 1}} d_{\lambda,\mu}$$

$$= \alpha n^v \sum_{\substack{|\lambda|+|\mu|=n \\ h(\lambda)\leqslant 2,\, \mu_1 \leqslant 1}} \binom{n}{|\lambda|} d_\lambda d_\mu \leqslant \alpha n^{v+1} \sum_{\substack{\lambda \vdash n \\ h(\lambda)\leqslant 2}} d_\lambda \leqslant \alpha' n^{v'} 2^n,$$

where the last inequality is deduced from the hook formula for the degrees of the irreducible representations of the symmetric group (see [4]). This proves (1).

The second part of the corollary is obtained by a direct calculation by making use of Theorem 3. □

## 6. Growth of the graded codimensions

In this section, we shall be involved with superalgebras and the growth of their codimensions. If $A$ is a superalgebra the growth of the sequence of graded codimensions of $A$ is called the $PI$-growth of $A$. We say that $A$ has almost polynomial $PI$-growth if $A$ has exponential growth but for every $T_2$-ideal $I \underset{\neq}{\supset} Id^{\mathrm{gr}}(UT_2)$ the sequence of graded codimensions of $I$ is polynomially bounded.

In Theorem 4 we shall prove that $UT_2$ with the canonical $\mathbf{Z}_2$-grading, is a superalgebra with almost polynomial $PI$-growth.

**Theorem 4.** *LetA be a $\mathbf{Z}_2$-graded algebra. If $Id^{\mathrm{gr}}(A) \underset{\neq}{\supset} Id^{\mathrm{gr}}(UT_2)$ then there exists a constant $N$ such that for all $n$ and $|\lambda|+|\mu|=n$ we have that $m_{\lambda,\mu}(A) \leqslant N$. Moreover $c_n^{\mathrm{gr}}(A)$ is polynomially bounded.*

**Proof.** Since $Id^{\mathrm{gr}}(A) \underset{\neq}{\supset} Id^{\mathrm{gr}}(UT_2)$, there exist $\lambda, \mu$ with $|\lambda|+|\mu|=n$ such that $m_{\lambda,\mu}(A) < m_{\lambda,\mu}(UT_2)$. It follows that the polynomials $a_{p,q}^{(i)}$ introduced in the proof of Theorem 3 are linearly dependent modulo $Id^{\mathrm{gr}}(UT_2)$. Hence

$$\sum_i \beta_i y_1^i \bar{y}_1 \cdots \tilde{y}_1 z \bar{y}_2 \cdots \tilde{y}_2 y_1^{q-i} = 0 \,(\mathrm{mod}\ Id^{\mathrm{gr}}(UT_2)).$$

Let $t = \max\{i : \beta_i \neq 0\}$. By substituting $y_2$ with $y_1^2$, we obtain

$$f(y_1,z) = \beta_t y_1^t \bar{y}_1 \cdots \tilde{y}_1 z \overline{(y_1)^2} \cdots \widetilde{(y_1)^2} y_1^{q-t}$$

$$+ \sum_{i<t} \beta_i y_1^i \bar{y}_1 \cdots \tilde{y}_1 z \overline{(y_1)^2} \cdots \widetilde{(y_1)^2} y_1^{q-i} = 0 \,(\mathrm{mod}\ Id^{\mathrm{gr}}(UT_2)).$$

Let $N = 3p + q = \deg(f(y_1,z)) - 1$ and $M = t + 2p$. Then, from the above, it follows that

$$y_1^M x y_1^{N-M} = \sum_{i<M} \delta_i y_1^i z y_1^{N-i} \,(\mathrm{mod}\ Id^{\mathrm{gr}}(UT_2)) \qquad (1)$$

for some $\delta_i \in F$.

We shall prove that $m_{\lambda,\mu}(A) \leqslant N$, for any partitions $\lambda, \mu,$. Clearly, by Theorem 3, it is enough to consider the case when $\lambda = (p+q, p)$, $\mu = (1)$. Notice that

$$\underbrace{\hat{y}_1 \cdots \tilde{y}_1}_{p} z \underbrace{\hat{y}_2 \cdots \tilde{y}_2}_{p} \in \mathscr{F}_1.$$

Hence, if $q \geqslant N$ we can apply (1) to any polynomial $a^{(i)}_{p,q}(y_1, y_2, z)$ as soon as $i \geqslant M$. We obtain

$$a^{(i)}_{p,q} = \sum_{j<M} \delta_j a^{(j)}_{p,q} \,(\text{mod } Id^{\text{gr}}(UT_2))$$

and $m_{\lambda,\mu}(A) \leqslant M \leqslant N$ follows.

We next show that the sequence $c^{\text{gr}}_n(A)$, $n = 1, 2, \ldots$, is polynomially bounded. By multilinearizing (1) we obtain

$$\sum_{\sigma \in S_N} y_{1\sigma(1)} \cdots y_{1\sigma(M)} z y_{1\sigma(M+1)} \cdots y_{1\sigma(N)}$$

$$= \sum_{i<M} \sum_{\sigma \in S_n} \delta_i y_{1\sigma(1)} \cdots y_{1\sigma(i)} z y_{1\sigma(i+1)} \cdots y_{1\sigma(N)} \,(\text{mod } Id^{\text{gr}}(UT_2)).$$

We multiply the above expression on the right by $y_{21} \cdots y_{2M}$ and we then alternate $y_{1i}$ with $y_{2i}$ for $i = 1, \ldots, M$. As a result we obtain

$$\bar{y}_{11} \hat{y}_{12} \cdots \tilde{y}_{1M} z \bar{y}_{21} \hat{y}_{22} \cdots \tilde{y}_{2M} y_{1\,M+1} \cdots y_{1N} = 0 \,(\text{mod } Id^{\text{gr}}(UT_2)).$$

If we now multiply on the left by $y_{2\,M+1} \cdots y_{2N}$ and then we alternate $y_{1j}$ with $y_{2j}$ for all $j = M+1, \ldots, N$ we get

$$\bar{y}_{11} \hat{y}_{12} \cdots \tilde{y}_{1N} z \bar{y}_{21} \hat{y}_{22} \cdots \tilde{y}_{2N} = 0 \,(\text{mod } Id^{\text{gr}}(UT_2)).$$

This relation shows that the irreducible $S_{N^2} \times S_1$-character corresponding to the pair of partitions $\lambda = (N^2), \mu = (1)$ participates into the $(N^2+1)$th graded cocharacter of $A$ with zero multiplicity, i.e., $m_{((N^2),(1))}(A) = 0$.

It follows that if $\lambda$ is a partition of $n$ such that $\lambda_2 \geqslant N$ then $m_{\lambda,\mu}(A) = 0$. Recalling Theorem 2, it follows that if $\chi_{\lambda,\mu}$ participates in the graded cocharacter with non-zero multiplicity then $\lambda$ must contain at most $N-1$ boxes below the first row and $|\mu| \leqslant 1$. Therefore

$$\chi^{\text{gr}}_n(A) = \sum_{\substack{|\lambda|+|\mu|=n \\ |\lambda|-\lambda_1 \leqslant N-1 \\ |\mu| \leqslant 1}} m_{\lambda,\mu} \chi_{\lambda,\mu}$$

and, by [2], $A$ has polynomial growth. $\square$

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